Energy input and scaling laws for a single particle vibrating in one dimension

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The one-dimensional motion of a single particle on a vibrating base is considered in the limit of high excitation (vibration frequency $\gg$ collision rate). An exact expression for the time averaged rate of energy input from the vibrating base to the particle is derived. By assuming a Gaussian form for the particle velocity distribution function, the expression can be numerically evaluated to obtain the one particle granular temperature as a function of the base velocity V and particle-base restitution coefficient ε. The granular temperature is shown to scale as V^2 and to scale approximately as $(1-\varepsilon)^{-1}$. The velocity scaling is also shown to hold over a generic class of velocity distribution functions. Assuming sinusoidal excitation yields scaling behavior identical to the sawtooth excitations used in the analysis, two different stable states can exist [(i) particle bouncing and (ii) particle not bouncing] when the peak base acceleration is less than g.

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I. INTRODUCTION

Granular materials subjected to vertical vibration exhibit a wide range of unusual behavior [1,2]. In model two-dimensional systems where high restitution coefficient particles are vertically vibrated, either surface fluidization [3] or almost total fluidization [4,5] can be observed. Total fluidization is confined to small system sizes and experimental observations, based on high speed photography and computer image processing [6], indicate that thermodynamic and kinetic theory concepts should be applicable, at least as a first approximation, to such granular assemblies [4,5]. The scaling behavior for the height of the center of mass in two dimensions has been obtained by computer simulation [5,7] and experimentally [4]; in the latter case the scaling for the system granular temperature was also investigated. Effort has therefore been made to gain an understanding of such fluidized granular materials by applying simplified rather than a rigorous kinetic theory of granular materials [8–11].

In an elementary treatment two main issues need to be considered. Firstly, the vibrating base acts as an energy source and the energy input to the particles as a function of base velocity and restitution coefficient needs to be evaluated. Secondly, particle-particle collisions are inelastic and therefore act as a source of energy dissipation. Calculation of the rate of energy input and rate of energy dissipation then allows one to determine the steady-state behavior of the system from an energy balance. Such a calculation has been performed in a highly fluidized two-dimensional system [4] where approximations were made so that a simple solution could be obtained. Similar scaling laws have also been obtained, for the almost condensed phase, in two dimensions by solving a Navier-Stokes type equation [5].

We consider here the simpler problem of a single particle undergoing one-dimensional (1D) vertical vibration. Previous work on the vibration of a single particle has concentrated on the rich phase behavior of the system which shows transitions to chaos [12–16]. These studies apply over a range of restitution coefficient ε and base acceleration Γ. It is found that the time for a cycle to repeat itself diverges as the power law $(1-\varepsilon)^{-5}$ for $\varepsilon \to 1$ and as Γ as $\Gamma \to \infty$. A one-dimensional column of beads has been shown to exhibit a transition from a condensed (high dissipation) to a fluidized (low dissipation) regime [17,18] and an analysis based on the dissipative Boltzmann equation, in the limit of low dissipation [19], has derived expressions which can be evaluated numerically to obtain density profiles and velocity distribution functions.

In this paper we derive an exact expression for the rate of energy input into the system. By assuming a form for the velocity distribution function of the particle we can numerically evaluate the expression and obtain the scaling laws for the particle granular temperature with base velocity and restitution coefficient for particle-base collisions. An understanding of this one-dimensional problem should then also be relevant to the related two-dimensional system in the low density fluidized regime.

II. SCALING ANALYSIS

A. Model

We consider a simple analysis which attempts to derive the scaling relationship, in a one-dimensional system, for the granular temperature as a function of the base velocity $A_0\omega$ and the restitution coefficient ε for particle-base collisions. In the spirit of our previous work [4], we consider the rate at which energy is put into the system by making the following initial assumptions. The sinusoidal oscillating base, amplitude A_0 and angular frequency ω, is approximated by a sawtooth with amplitude sufficiently small that the velocity of a particle coming down will not change appreciably due to the effect of gravity over a dis-

tance equal to the amplitude. We further assume that the frequency of the base is so high that the particle coming down sees the base moving at a velocity which is uncorrelated with the value on previous collisions. This seems a reasonable assumption provided that the excitation is sufficiently strong to ensure that the mean time between collisions is much greater than the vibration period. The time at which the particle passes the origin of the vertical axis on its way down can therefore be regarded as a random variable with uniform probability distribution over the period of the base.

We extend our previous analysis [4] by identifying all possible collision events together with their probabilities of occurrence and the energy changes associated with them. In the laboratory reference frame, we denote the precollisional particle velocity and the base velocity by v_p and v_b, respectively. In general, the definition of the restitution coefficient leads to a postcollisional velocity for the particle of $(1+\varepsilon)v_b-\varepsilon v_p$ and there are two initial cases that need to be considered. Defining upwards as positive, we denote the two situations by case I and case II defined with the base moving upwards or downwards, respectively.

We let P_1 and P_2 denote the probabilities of cases I and II occurring in a given collision. Considering a particle moving down towards the base then we denote the particle velocity by $v_p=-v$ (where $v\geq 0$), and the base velocity by $v_b=\pm V$ (where $V\geq 0$). If $v\leq V$ the particle will never catch the base on the way down and only a case I collision will result. For $v>V$ both case I and case II collisions can occur. Referring to the position time diagram in Fig. 1, the probabilities of such a case I and case II collision occurring can be calculated. If the particle passes through the line $y=0$ in region $A-B$ then case I applies otherwise case II applies; the relative probabilities only depend on the lengths L_{OA} and L_{AB}. From the geometry of the figure we have

$$L_{AB}=L_{AC}+L_{CB}=\frac{T}{2}\left[1+\frac{V}{v}\right]$$

and

$$L_{OA}=L_{OB}-L_{AB}=\frac{T}{2}\left[1-\frac{V}{v}\right],$$

so that the probabilities, P_1 and P_2 are given by the ratios L_{AB}/L_{OB} and L_{OA}/L_{OB}, respectively, resulting in

$$P_1=\begin{cases}1\ , & v\leq V\\ \frac{1}{2}\left[1+\frac{V}{v}\right]\ , & v\geq V\end{cases} \tag{1}$$

and

$$P_2=\begin{cases}0\ , & v\leq V\\ \frac{1}{2}\left[1-\frac{V}{v}\right]\ , & v\geq V\ .\end{cases} \tag{2}$$

After a case I collision, the particle velocity is $\varepsilon v+(1+\varepsilon)V$. After a case II collision, the particle velocity is $\varepsilon v-(1+\varepsilon)V$ and if this postcollisional velocity is negative (particle still moving down) then a second impact will always occur. The condition for this is $V\leq v\leq(1+1/\varepsilon)V$. Conversely if the initial postcollisional velocity is positive and greater than V only the single impact will occur. The condition for this is $v\geq(1+2/\varepsilon)V$. However, for intermediate preimpact velocities $[(1+1/\varepsilon)V\leq v\leq(1+2/\varepsilon)V]$ the postimpact velocity will be positive but less than V and either a single or a double collision can occur. Since the initial postcollisional velocity after the first case II collision is $v_2=\varepsilon v-(1+\varepsilon)V$, the velocity after a second collision is $(1+\varepsilon)^2V-\varepsilon^2 v$.

When a case II collision takes place we let P_3 and P_4, respectively, denote the probability that a second collision with the base, after it changes direction and moves upwards again, either does not or does take place. The intermediate velocity range $(1+1/\varepsilon)V\leq v\leq(1+2/\varepsilon)V$ is considered in Fig. 2. If the initial case II collision takes place in the projected region L_{FG} then a double collision occurs, whereas if it takes place in projected region L_{GH} then only this initial single collision will occur. From the geometry of Fig. 2 we find that a double (case IV) collision takes place if $-[VT/\varepsilon(v-V)]\leq t\leq -T/2$ and a single (case III) collision takes place if $-T\leq t\leq -[VT/\varepsilon(v-V)]$. The probabilities P_3 and P_4, which are valid in the interval $(1+1/\varepsilon)V\leq v\leq(1+2/\varepsilon)V$, are then given by the ratios L_{GH}/L_{FH}

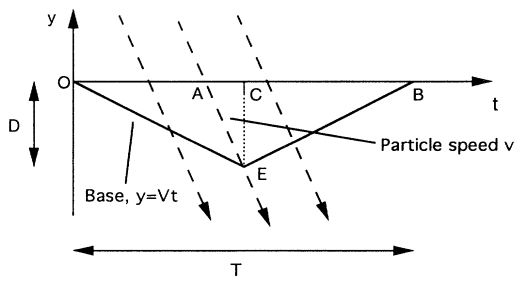

FIG. 1. Distance time graph for a particle approaching the sawtooth base. The probabilities P_1 and P_2 are calculated from this plot as discussed in the text.

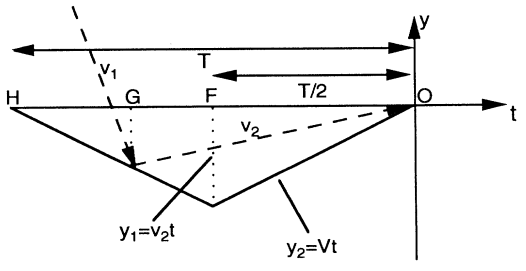

FIG. 2. Distance time graph for particle trajectories after the first collision with the base moving downwards. The probabilities P_3 and P_4 are calculated from this plot as discussed in the text.

and L_{FG}/L_{FH}, respectively. Including the trivial cases we therefore have that

$$P_3=\begin{cases}0\ , & V\le v\le(1+1/\varepsilon)V\\ \dfrac{2[\varepsilon(v-V)-V]}{\varepsilon(v-V)}\ , & (1+1/\varepsilon)V\le v\le(1+2/\varepsilon)V\\ 1\ , & (1+2/\varepsilon)V\le v\le\infty\end{cases} \tag{3}$$

and

$$P_4=\begin{cases}1\ , & V\le v\le(1+1/\varepsilon)V\\ \dfrac{[\varepsilon(V-v)+2V]}{\varepsilon(v-V)}\ , & (1+1/\varepsilon)V\le v\le(1+2/\varepsilon)V\\ 0\ , & (1+2/\varepsilon)V\le v\le\infty\ .\end{cases} \tag{4}$$

In the interval $(1+1/\varepsilon)V\le v\le(1+2/\varepsilon)V$ the total probability of a double and single collision are given by P_2P_4 and P_2P_3, respectively.

Given the above probability model and the velocity changes associated with the various collision events we can now calculate the corresponding energy changes. For a simple case I collision, the energy change is given by

$$\Delta E_1=\frac{m}{2}[(\varepsilon^2-1)v^2+2\varepsilon(1+\varepsilon)vV+(1+\varepsilon)^2V^2]\ . \tag{5}$$

In the regime after a case II collision two energy change equations are required given by

$$\Delta E_2^s=\frac{m}{2}[(\varepsilon^2-1)v^2-2\varepsilon(1+\varepsilon)vV+(1+\varepsilon)^2V^2] \tag{6}$$

and

$$\Delta E_2^d=\frac{m}{2}[(\varepsilon^4-1)v^2-2(1+\varepsilon)^2\varepsilon^2vV+(1+\varepsilon)^4V^2]\ , \tag{7}$$

where the superscripts s and d refer to a single and double collision, respectively.

The various cases are summarized in Fig. 3. Given the probability of the occurrence of the various collision events and the kinetic energy changes associated with them we can now proceed to calculate an exact expression for the net rate of energy transfer to the particle from the base. It is easier from a conceptual viewpoint (although not necessary from a mathematical one) to imagine a large set of equivalent but uncorrelated and noninteracting bouncing particles. This then allows straightforward application of standard gas kinetic theory, though in 1D rather than three dimensions (3D). If the instantaneous velocity distribution function and number density at $y=0$ for the ensemble are $f(v)$ and n_0, respectively, then the number of particles passing through $y=0$ per unit time with velocities in the range v to $v+dv$ is $n_0vf(v)dv$. The net rate of energy transfer to the particles from the base is then given by

$$\begin{aligned}\dot{E}=n_0\Big\{&\int_0^V\Delta E_1f(v)v\,dv+\int_V^\infty\Delta E_1P_1f(v)v\,dv\\ &+\int_V^{(1+1/\varepsilon)V}\Delta E_2^dP_2f(v)v\,dv\\ &+\int_{(1+1/\varepsilon)V}^{(1+2/\varepsilon)V}\Delta E_2^sP_2P_3f(v)v\,dv\\ &+\int_{(1+1/\varepsilon)V}^{(1+2/\varepsilon)V}\Delta E_2^dP_2P_4f(v)v\,dv\\ &+\int_{(1+2/\varepsilon)V}^\infty\Delta E_2^sP_2f(v)v\,dv\Big\}\ ,\end{aligned} \tag{8}$$

where P_1, P_2, P_3, and P_4 are given in Eqs. (1)–(4) and ΔE_1, ΔE_2^s, and ΔE_2^d are given in Eqs. (5)–(7). If $f(v)$ is a very narrow distribution (low speed particles) then only the first term will contribute; since this is positive the ensemble will gain energy from the base, causing $f(v)$ to broaden. If, on the other hand, $f(v)$ is very broad (many high speed particles) then the second and last terms will dominate (with P_1 and $P_2\approx\frac{1}{2}$). The sum of these two terms is negative so energy is extracted from the parti-

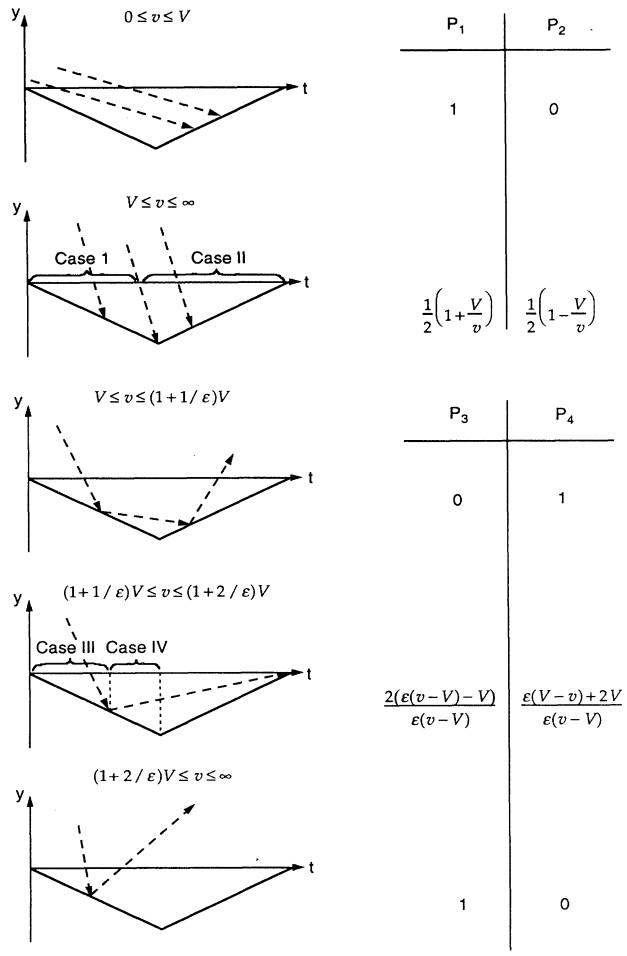

FIG. 3. Summary of the various collision cases and their associated probabilities and energy changes. P_1 is given by Eq. (1) for all $v>V$. The various velocity ranges and energy changes are calculated in the text.

cles, on average, causing $f(v)$ to narrow. We therefore expect some equilibrium $f(v)$ and n_0 to be set up such that $\dot{E}=0$.

To proceed further, we assume that $f(v)$ takes the usual Gaussian form from gas kinetic theory

$$f(v)=\left[\frac{m}{2\pi E_0}\right]^{1/2}\exp\left[-\frac{m}{2E_0}v^2\right], \tag{9}$$

where E_0 is the granular temperature of the particle. The integrals can be simplified by defining two-dimensionless variables

$$\beta=\frac{v}{V} \tag{10}$$

and

$$\delta=\frac{U}{V}, \tag{11}$$

where the only new variable introduced U is the width of the Gaussian velocity distribution, that is $U=\sqrt{E_0/m}$. Making the substitutions of Eqs. (10) and (11) in Eq. (8) then leads to

$$\begin{aligned}\dot{E}=A\Bigg\{&2\int_0^1[(\varepsilon^2-1)\beta^3+2\varepsilon(1+\varepsilon)\beta^2+(1+\varepsilon)^2\beta]\exp\left[-\frac{\beta^2}{2\delta^2}\right]d\beta\\&+\int_1^{\infty}[(\varepsilon^2-1)\beta^2+2\varepsilon(1+\varepsilon)\beta+(1+\varepsilon)^2](1+\beta)\exp\left[-\frac{\beta^2}{2\delta^2}\right]d\beta\\&+\int_1^{1+1/\varepsilon}[(\varepsilon^4-1)\beta^2-2(1+\varepsilon)^2\varepsilon^2\beta+(1+\varepsilon)^4](\beta-1)\exp\left[-\frac{\beta^2}{2\delta^2}\right]d\beta\\&+2\int_{1+1/\varepsilon}^{1+2/\varepsilon}[(\varepsilon^2-1)\beta^2-2\varepsilon(1+\varepsilon)\beta+(1+\varepsilon)^2][(\beta-1)-1/\varepsilon]\exp\left[-\frac{\beta^2}{2\delta^2}\right]d\beta\\&+\int_{1+1/\varepsilon}^{1+2/\varepsilon}[(\varepsilon^4-1)\beta^2-2(1+\varepsilon)^2\varepsilon^2\beta+(1+\varepsilon)^4][(1-\beta)+2/\varepsilon]\exp\left[-\frac{\beta^2}{2\delta^2}\right]d\beta\\&+\int_{1+2/\varepsilon}^{\infty}[(\varepsilon^2-1)\beta^2-2\varepsilon(1+\varepsilon)\beta+(1+\varepsilon)^2](\beta-1)\exp\left[-\frac{\beta^2}{2\delta^2}\right]d\beta\Bigg\},\end{aligned} \tag{12}$$

where A is given by

$$A=\frac{mV^4n_0(U)}{4\sqrt{2\pi}U}. \tag{13}$$

We next consider how to extract the scaling law for the particle granular temperature E_0 with the base velocity V and particle-base restitution coefficient ε. For each value of V or ε one can take a value for the granular temperature (contained within the term U) and calculate the integral in Eq. (12) to obtain the rate of energy transfer from the base to the particle. By selecting the appropriate granular temperature which gives zero rate of energy transfer, the equilibrium condition established can be obtained. The integrals in Eq. (12) are functions only of β and δ and the prefactor A can never be zero (because we neglect the cases $V=0$ and $U=$ infinity). Zero rate of energy transfer therefore requires the total integral to be zero. For a fixed ε, this will happen at a particular value of δ and any combination of U and V which have this ratio will make the rate of energy transfer zero and hence determine the steady-state granular temperature that would be produced. Thus U must scale in proportion to V and the granular temperature scale as V^2. We can therefore write

$$E_0=g(\varepsilon)V^2. \tag{14}$$

In order to calculate the value of δ which makes $\dot{E}=0$ for each value of ε and thus to determine the function $g(\varepsilon)$ we evaluated Eq. (12) numerically, using Numerical Algorithm Group (NAG) library routines d01akf and d01amf. The total integral was calculated as a function of the granular temperature and a simple interval bisection routine used to find the granular temperature that gives a zero rate of energy transfer. Only a single root was found to exist over the range of ε studied.

It is worth pointing out that the scaling of E_0 with V [Eq. (14)] does not necessarily require $f(v)$ to be Gaussian, but will hold provided $f(v)$ can be expressed in the form

$$f(v,U)=U^{-1}h(v/U), \tag{15}$$

where U is now some characteristic width of the distribution. Equation (15) is satisfied if the shape of the distribution, after normalization of the velocity axis by U, does not change with U.

B. Results and discussion

Figure 4 shows a nondimensional plot of δ versus the restitution coefficient ε. As ε increases, the steady-state characteristic grain velocity U becomes steadily larger than the base velocity V. There is a divergence to infinity when epsilon equals unity which corresponds to a continual increase of granular temperature with time because

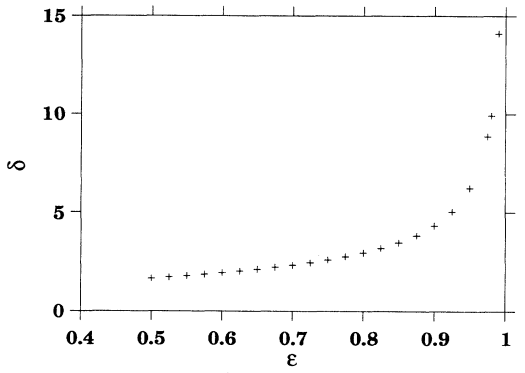

FIG. 4. Nondimensional plot of δ, defined by Eq. (14), against ε, the particle-base restitution coefficient. δ governs the magnitude of E_0 relative to the base velocity V.

no energy is dissipated in particle-base collisions. The ε dependence is replotted in Fig. 5, where δ is shown against $(1-\varepsilon)$. This indicates quite a close approximation to power law behavior, though with some deviation below around $\varepsilon=0.7$; masking these points out allows a power law exponent of -0.52 to be fitted to all the data. Figure (5) therefore shows, to a good approximation, that

$$\frac{U}{V}=1.3(1-\varepsilon)^{-0.52}\ , \tag{16}$$

and thus

$$E_0=mU^2=1.7\frac{mV^2}{(1-\varepsilon)}\ . \tag{17}$$

The scaling laws derived for the granular temperature of a single particle are identical to those found for a one-dimensional column of particles in the fluidized regime [17]. In the limit of low density, these results should also govern the scaling behavior in two-dimensional systems.

Although the analysis presented here used a sawtooth displacement-time function for the base, it is reasonable to assume that the same scaling dependence will arise with sinusoidal excitation at the same angular frequency ω and with amplitude A_0 such that the peak velocity also equals V (i.e., $A_0=V/\omega$). If this is the case, one interesting consequence of the analysis is that "fluidization" is sustainable even for peak base acceleration less than g. This follows from the requirement that the time between contacts is much greater than the driving period, i.e., that

$$\frac{2U}{g}\gg\frac{2\pi}{\omega}\ , \tag{18}$$

and hence that

$$\frac{A_0\omega^2}{g}\gg 2.4(1-\varepsilon)^{1/2}\ . \tag{19}$$

The analysis presented here (sustainable bouncing on the base) is therefore valid even if the left hand side of Eq. (19) is less than unity, provided the collisions are sufficiently elastic. On the other hand, a grain initially at rest on a stationary base will remain in contact with the base until the peak acceleration $A_0\omega^2$ reaches g. This implies that there are two stable states for the system when $A_0\omega^2<g$, possessing different mean center-of-mass heights, and the actual state adopted depends on the past history of the system.

Finally, we note that the form of the velocity distribution function, when velocities are normalized by V, is independent of the magnitude of the base velocity. This is because the collision probabilities and postcollisional velocities are functions only of the normalized velocities. Thus if we imagine a set of velocities produced by a set of bounces and now double the base velocity V and the initial velocity before the first bounce, then the subsequent set of velocities will also be exactly doubled in magnitude and therefore the resulting distribution function is unchanged when the velocities are normalized. Experiments are currently in progress to measure the velocity distribution function for a one-dimensional bouncing ball from which the granular temperature can be extracted. A frequency and vibration amplitude A_0 of 100 Hz and upto 3/4 mm, respectively, are found to be realistic experimental conditions for satisfying the assumptions in the model.

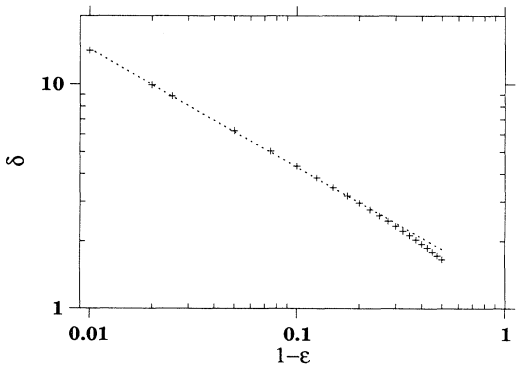

FIG. 5. Results from Fig. 4 replotted to show variation of δ with $(1-\varepsilon)$.

III. CONCLUSIONS

An exact expression for the time averaged rate of energy input to a single particle undergoing one-dimensional vertical vibration has been derived. By assuming a Gaussian velocity distribution we have numerically calculated the granular temperature as a function of base velocity V and particle-base restitution coefficient ε. The granular temperature is shown to scale as V^2 and to scale as $(1-\varepsilon)^{-1}$. If it is assumed that sinusoidal excitation produces the same scaling laws as sawtooth excitations then the system exhibits bistable behavior for highly elastic impacts when the peak acceleration is less than g. Our result is based on elementary kinetic theory concepts and should provide a useful starting point for developing a more complete theory of granular materials.

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